and a parallel side of the drilled square hole is $\frac{a-x}{2}$.

Now consider the surface area "inside" the cube made by the part of the drilled square that starts at a side of the original cube and ends when the drilled square meets the other drilled square originating from an adjacent side of the cube. This surface area looking at one side of the cube includes four rectangles with one side length of x and "depth" length of $\frac{a-x}{2}$, so this surface area is$\frac{4x(a-x)}{2} = 2(a-x)$. There are four of these around the original cube. The surface area of each of the two sides of the original cube which have no holes is a.

In the middle of the original cube at the intersection of the two drilled square holes, there are two squares of side length x with are parallel to the sides of the original cube with no holes . The area of each square is x^2.

The total surface area of the problem is

$$4(a^2-x^2)+4\left(2x(a-x)\right)+2a^2+2x^2 = 6a^2+8ax-10x^2.$$

The maximum surface area occurs when $8a-20x=0$ or $x=\frac{2a}{5}$. The maximum surface area is $\frac{38a^2}{5}$ when a side of the drilled square holes as a length of $\frac{2a}{5}$.

Editor′s comment: **David Stone and John Hawkins, both from Georgia Southern University, Statesboro, GA** accompanied their solution by placing the statement of the problem into a story setting. They wrote:

"An interpretation: in the ancient Martian civilization, the rulers favorite meditational spot was a levitating cube having a cubical inner sanctum formed by two horizontal square tunnels, meeting at the center of the cube, from which he could see out in all four directions. The designers were charged to construct the ship with a maximum amount of wall space for inscriptions and carved likenesses of His Highness. There are four short hallways leading from the inner room to the outside walls." They let x be the side length of the square tunnels that are drilled through the original cube and noted that each tunnel has an $x \times x$ cross section and has length a. The inner most cubical room is $x \times x \times x$. They then mentioned that "by drilling the tunnels and opening up an interior chamber, the surface area has increased from $6a^2$ to $\frac{38}{5}a^2$, an increase of $\frac{8}{5}a^2$ or 27%. So the King has his private getaway and more space for pictures and wall hangings."

Also solved by Jeremiah Bartz, University of North Dakota, Grand Forks, ND and Nicholas Newman, Francis Marion University, Florence SC; Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel; David A. Huckaby, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5417:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any positive real number x, and for any natural number $n \geq 2$,

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}}.$$

Solution 1 by Henry Ricardo, New York Math Circle, NY

Let $\alpha_n = (1+x+\cdots+x^n)/(n+1)$ and define

$$F(x) = \frac{(1+x+x^2+\cdots+x^{n-1})^n}{(1+x+x^2+\cdots+x^n)^{n-1}}.$$

Then, for $x>0$ and $n \ge 2$, we see that

$$\sqrt[n-1]{\alpha_{n-1}} \le \sqrt[n]{\alpha_n} \;\Leftrightarrow\; \alpha_{n-1}^n \le \alpha_n^{n-1} \;\Leftrightarrow\; F(x) \le \frac{n^n}{(n+1)^{n-1}} = F(1).$$

Now we show that $F(x)$ attains its absolute maximum value at $x=1$.

For $x \ne 1$, we have

$$\begin{aligned} F'(x) &= \frac{(x^n-1)^{n-1}(x^{n+1}-1)^{-n}\left(-x^{2n+1}+n^2x^{n+2}+2(1-n^2)x^{n+1}+n^2x^n-x\right)}{x(x-1)^2} \\ &= \overbrace{\frac{(x^n-1)^{n-1}}{(x^{n+1}-1)^n(x-1)^2}}^{G(x)} \cdot \overbrace{\left(-x^{2n}+n^2x^{n+1}+2(1-n^2)x^n+n^2x^{n-1}-1\right)}^{H(x)}. \end{aligned}$$

Noting that $G(x)$ is negative for $0<x<1$ and positive for $x>1$, we examine the factor $H(x)$ to see that

$$\begin{aligned} H(x) &= -(x^n-1)^2 + n^2x^{n-1}(x-1)^2 \\ &= -n^2(x-1)^2\left[\frac{(x^{n-1}+x^{n-2}+\cdots+x+1)^2}{n^2} - x^{n-1}\right] \\ &= -n^2(x-1)^2\left[\left(\frac{x^{n-1}+x^{n-2}+\cdots+x+1}{n}\right)^2 - \left(\sqrt[n]{x^{n-1}\cdot x^{n-2}\cdots x\cdot 1}\right)^2\right] \end{aligned}$$

is negative for all $x>0$ by the AM-GM inequality.

Thus $F'(x)>0$ for $0<x<1$ and $F'(x)<0$ for $x>1$, implying that $F(x)$ has an absolute maximum value at $x=1$—that is, $F(x) \le F(1)$ on $(0,\infty)$, which proves the proposed inequality.

COMMENT: This was proposed by Walther Janous as problem 1763 (1992, p. 206) in *Crux Mathematicorum*. My solution is based on the published solution of Chris Wildhagen.

Solution 2: by Moti Levy, Rehovot, Israel

If $x=1$ then the inequality holds, since

$$\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} = \sqrt[n-1]{\frac{1+x+\cdots+x^{n-1}}{n}} = 1.$$

We assume that $x>1$.
Let us define the continuous functions $g(t)$, and $f(t)$, $t \in R$, $t>1$, as follows,

$$g(t) := \frac{x^{t+1}-1}{x-1}\frac{1}{t+1}, \quad f(t) := (g(t))^{\frac{1}{t}} .$$

Clearly, $\sqrt[n]{\frac{1+x+\cdots+x^n}{n+1}} = \sqrt[n]{\frac{1}{n+1}\frac{x^{n+1}-1}{x-1}} = f(n)$. The original inequality (in terms of the function f) is

$$f(n) \geq f(n-1), \quad \text{for} \quad n \geq 2.$$

For $n=2$, $\sqrt{\frac{1+x+x^2}{3}} \geq \frac{1+x}{2}$ follows from $\frac{1+x+x^2}{3} - \left(\frac{1+x}{2}\right)^2 = \frac{1}{12}(x-1)^2 \geq 0$.
Therefore, it suffices to prove that $f(t)$ is monotone increasing function for $t \geq 1$.
We will show this by proving that the derivative of $\ln f(t)$ is postive for $t \geq 1$.
The derivative is given by

$$t^2 \frac{d}{dt}(\ln f) = -\ln g + t\frac{\frac{dg}{dt}}{g}.$$

The first step is showing $-\ln g + t\frac{\frac{dg}{dt}}{g} > 0$ for $t = 1$.

$$-\ln g + t\frac{\frac{dg}{dt}}{g}\bigg|_{t=1} = -\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}.$$

To show that $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)} > 0$ for $x > 0$, we see that $\lim_{x\to 0}\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right) = \ln 4 > 0$.
Now we show that the derivative of $-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}$ is positive:

$$\frac{d\left(-\ln\left(\frac{1+x}{4}\right) + \frac{2x^2\ln x}{2(x^2-1)}\right)}{dx} = \frac{1}{x^2-1} - \frac{2x\ln x}{(x^2-1)^2}.$$

We use the well known inequality: $\ln x \leq \frac{x^2-1}{2x}$ for $x > 0$ to show that

$$\frac{1}{x^2-1} - \frac{2x\ln x}{(x^2-1)^2} \geq 0.$$

The second step is showing that the derivative of $-\ln g + t\frac{\frac{dg}{dt}}{g}$ is positive for $t > 0$,

$$\frac{d\left(-\ln g + t\frac{\frac{dg}{dt}}{g}\right)}{dt} = -\frac{\frac{dg}{dt}}{g} + \frac{\frac{dg}{dt}}{g} + \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right).$$

After some tedious calculation we arrive at,

$$\frac{d}{dt}\left(\frac{\frac{dg}{dt}}{g}\right) = \frac{\left(x^{t+1}-1\right)^2 - x^{t+1}\ln^2 x^{t+1}}{\left(x^{t+1}-1\right)^2(t+1)^2}.$$

To show that $\left(x^{t+1}-1\right)^2 \geq x^{t+1}\ln^2 x^{t+1}$, or that $\ln x^{t+1} \leq \frac{1}{\sqrt{x^{t+1}}}\left(x^{t+1}-1\right)$, we use again the inequality $\ln y \leq \frac{y^2-1}{2y}$ for $y > 0$,

$$\ln y \leq \frac{y-1}{\sqrt{y}}\frac{y+1}{2\sqrt{y}}.$$

But $\frac{y+1}{2\sqrt{y}} \geq 1$; hence,

$$\ln y \leq \frac{y-1}{\sqrt{y}}, \quad y > 0.$$

Now set $y = x^{t+1}$ to finish the proof.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the inequality of the problem by $(*)$. It is easy to see that if $(*)$ holds for $x = t$ then it also holds for $x = \frac{1}{t}$. Hence it suffices to prove $(*)$ for $0 < x \leq 1$.

Let $f(x) = (n-1)\ln\left(\sum_{k=0}^{n} x^k\right) - n\ln\left(\sum_{k=0}^{n-1} x^k\right) + \ln\left(\frac{n^n}{(n+1)^{n-1}}\right)$, where $0 < x \leq 1$.

By taking logarithms, we see that $(*)$ is equivalent to $f(x) \geq 0$.

We have $f(1) = 0$ and for $0 < x < 1$,

$$f(x) = (n-1)\ln(1-x^{n+1}) - n\ln(1-x^n) + \ln(1-x) + \ln\left(\frac{n^n}{(n+1)^{n-1}}\right).$$

Hence to prove $(*)$, we need only prove that $f'(x) < 0$ for $0 < x < 1$.

Since $f'(x) = \frac{g(x)}{(x-1)(x^n-1)(x^{n+1}-1)}$, where

$g(x) = x^{2n} - n^2x^{n+1} + 2(n-1)(n+1)x^n - n^2x^{n-1} + 1$, it suffices to show

$g(x) > 0$, for $0 < x < 1$. Now

$$g'(x) = 2nx^{2n-1} - (n+1)n^2x^n + 2n(n-1)(n+1)x^{n-1} - (n-1)n^2x^{n-2},$$

$$g''(x) = 2n(2n-1)x^{2n-2} - (n+1)n^3x^{n-1} + 2n(n+1)(n-1)^2x^{n-2} - (n-1)(n-2)n^2x^{n-3}, \text{and}$$

$$g'''(x) = 4n(n-1)(2n-1)x^{2n-3} - (n-1)(*n+1)n^3xn - 2+$$
$$2n(n-2)(n+1)(n-1)^2x^{n-3} - (n-1)(n-2)(n-3)n^2x^{n-4}.$$

Thus $g(1) = g'(1) = g''(1) = g'''(x) = 0$ so that 1 is a root of multiplicity 4 of the equation $g(x) = 0$. By Descartes' rule of signs, the equation $g(x) = 0$ has no other positive roots. Since $g(0) = 1 > 0$, so $g(x) > 0$ for $0 < x < 1$.

This completes the proof.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $f(t) = 1/x$. The inequality goes unchanged because

$$\sqrt[n]{\frac{1+\frac{1}{t}+\ldots+\frac{1}{t^n}}{t^n(n+1)}} \geq \sqrt[n-1]{\frac{1+\frac{1}{t}+\ldots+\frac{1}{t^{n-1}}}{t^{n-1}n}}$$
$$\iff \sqrt[n]{\frac{1+t+\ldots+t^n}{n+1}} \geq \sqrt[n-1]{\frac{1+t+\ldots+t^{n-1}}{n}}.$$

This means that we may assume $x \geq 1$.

Let $x = 1$. The inequality becomes

$$1 = \sqrt[n]{\frac{1}{n+1}(\underbrace{1+1+\ldots+1}_{n+1 \ times})} \geq \sqrt[n]{\frac{1}{n}(\underbrace{1+1+\ldots+1}_{n \ times})} = 1.$$

Let $x > 1$. The inequality is also

$$\sqrt[n]{\frac{1}{n+1}\frac{1-x^{n+1}}{1-x}} \geq \sqrt[n-1]{\frac{1}{n}\frac{1-x^n}{1-x}},$$

that is

$$\sqrt[n]{\frac{1}{x-1}\int_1^x t^n} \geq \sqrt[n-1]{\frac{1}{x-1}\int_1^x t^{n-1}}.$$

This is the Power–Means inequality for integrals.

Also solved by Ed Gray, Highland Beach, FL; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5418:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

Let ABC be an acute triangle with circumradius R and inradius r. If $m \geq 0$, then prove that

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} \geq \frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 1 by Nikos Kalapodis, Patras, Greece

Applying Radon's Inequality and taking into account that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and $\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$ (see Solution 1 of Problem 5381, SSMA, April 2016) we have

$$\sum_{cyclic} \frac{\cos A \cos^{m+1} B}{\cos^{m+1} C} = \sum_{cyclic} \frac{\left(\dfrac{\cos A \cos B}{\cos C}\right)^{m+1}}{\cos^m A} \geq \frac{\left(\displaystyle\sum_{cyclic} \frac{\cos A \cos B}{\cos C}\right)^{m+1}}{\left(\displaystyle\sum_{cyclic} \cos A\right)^m} \geq$$

$$\frac{3^{m+1} R^m}{2^{m+1}(R+r)^m}.$$

Solution 2 by Arkady Alt, San Jose, CA

Firstly, we will prove that in any acute triangle the inequality

(1) $\displaystyle\sum_{cyc} \frac{\cos A \cos B}{\cos C} \geq \frac{3}{2}$, holds.